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Lagrange Interpolation for the Disk Algebra: The Worst Case

Gerd Herzog

Mathematisches Institut I, Universität Karlsruhe, D-76128 Karlsruhe, Germany
E-mail: Gerd.Herzog@math.uni-karlsruhe.de

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We consider Lagrange interpolation polynomials for functions in the disk algebra with nodes on the boundary of the unit disk. In case that the closure of the set of nodes does not cover the boundary of the unit disk we prove that there exists a residual set of functions in the disk algebra, such that the Lagrange interpolation polynomials of each of these functions form a dense subset of the space of all holomorphic functions defined on the unit disk.

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let $H(\mathbb{D})$ denote the Fréchet space of all holomorphic functions on $\mathbb{D}$ endowed with the compact open topology, and as usual let $A(\mathbb{D}) = H(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ be the disk algebra, normed with the maximum norm $\|\cdot\|_\infty$. Let $K = \{z_{kn}: 0 \leqslant k \leqslant n, n \in \mathbb{N}_0\}$ be a matrix of nodes in $\partial\mathbb{D}$, that is,

$$K = \begin{pmatrix} z_{00} & 0 & 0 & \cdots \\ z_{01} & z_{11} & 0 & \cdots \\ z_{02} & z_{12} & z_{22} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $|z_{kn}| = 1$ $(0 \leqslant k \leqslant n \in \mathbb{N}_0)$ and $z_{kn} \neq z_{jn}$ $(0 \leqslant k < j \leqslant n \in \mathbb{N}_0)$. For $n \in \mathbb{N}_0$ let $L_{K,n}: A(\mathbb{D}) \to H(\mathbb{D})$ denote the Lagrange interpolation operator

$$(L_{K,n} f)(z) = \sum_{k=0}^{n} f(z_{kn}) \frac{\omega_n(z)}{(z - z_{kn})\, \omega_n'(z_{kn})},$$

with $\omega_n(z) = \prod_{k=0}^{n} (z - z_{kn})$. For $0 < r < 1$ let $\|\cdot\|_r$ denote the norm $\|g\|_r = \max\{|g(z)|: |z| \leqslant r\}$ on $H(\mathbb{D})$, and note that for each sequence

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$(r_k)_{k=1}^{\infty}$ in (0, 1) with limit 1, the sequence of norms $(\|\cdot\|_{r_k})_{k=1}^{\infty}$ generates the compact open topology on $H(\mathbb{D})$.

Several authors have studied conditions for convergence and divergence of the sequence $(L_{K,n}f)_{n=0}^{\infty}$, see [1–5, 11, 12]. Recently Boche [1] has characterized those matrices K for which $L_{K,n}f \to f$ $(n \to \infty)$ in $H(\mathbb{D})$ for all $f \in A(\mathbb{D})$.

In this paper we are concerned with bad behaviour of Lagrange interpolation polynomials. We consider the set

$$U := \{f \in A(\mathbb{D}) : \{L_{K,n}f : n \in \mathbb{N}_0\} \text{ is dense in } H(\mathbb{D})\}.$$

If $f \in U$, the sequence of Lagrange interpolation polynomials of f is not only divergent; each function in $H(\mathbb{D})$ can be approximated by a subsequence of $(L_{K,n}f)_{n=0}^{\infty}$. We will use results on universal functions; see [6]. First note that both spaces, $A(\mathbb{D})$ and $H(\mathbb{D})$, are Baire spaces and separable (the polynomials are dense in both spaces), and that the mappings $L_{K,n}: A(\mathbb{D}) \to H(\mathbb{D})$ are continuous. Since $L_{K,n}p \to p$ $(n \to \infty)$ for each polynomial p, Proposition 6 in [6] implies that U is either empty or residual (that is, $A(\mathbb{D}) \setminus U$ is of first category). The following result contains a sufficient condition on K so that U becomes residual.

THEOREM 1. *Let $K = \{z_{kn} : 0 \leqslant k \leqslant n, n \in \mathbb{N}_0\}$ be a matrix of nodes whose closure is a proper subset of $\partial\mathbb{D}$. Then the set U of all functions $f \in A(\mathbb{D})$ for which $\{L_{K,n}f : n \in \mathbb{N}_0\}$ is dense in $H(\mathbb{D})$, is a residual subset of $A(\mathbb{D})$.*

By means of Theorem 1 it is possible to prove that there are even nodes with $\bar{K} = \partial\mathbb{D}$ and U residual.

THEOREM 2. *There is a matrix $K = \{z_{kn} : 0 \leqslant k \leqslant n, n \in \mathbb{N}_0\}$ of nodes in $\partial\mathbb{D}$ with*

$$\lim_{n \to \infty} \max \{|z_{1n} - z_{0n}|, \ldots, |z_{nn} - z_{n-1n}|, |z_{0n} - z_{nn}|\} = 0, \tag{1}$$

for which the set U is residual.

Remarks. (1) Condition (1) can be valid in cases in which $L_{K,n}f \to f$ in $H(\mathbb{D})$ $(n \to \infty)$ for each $f \in A(\mathbb{D})$; see [1]. Then $U = \varnothing$.

(2) Results related to Theorems 1 and 2 are known for Lagrange interpolation polynomials of real functions on an interval; see [7].

Proof of Theorem 1. Let $p \in A(\mathbb{D})$, $q \in H(\mathbb{D})$ be polynomials, let $\varepsilon > 0$ and $r \in (0, 1)$. According to the Universality Criterion, see [6, Theorem 1], we are done if we can prove:

There exists $f \in A(\mathbb{D})$ and $n_0 \in \mathbb{N}$ such that

$$\|f-p\|_\infty < \varepsilon \qquad \text{and} \qquad \|L_{K,n_0} f - q\|_r < \varepsilon.$$

Since $\bar{K} \neq \partial\mathbb{D}$ there is an interval $[a, b]$ with $0 < b-a < 2\pi$ such that $C = \{\exp(it) \colon t \in [a, b]\}$ contains $\bar{K}$. The set $C \cup (r\bar{\mathbb{D}})$ is a compact subset of $\mathbb{C}$ with connected complement.

Let $M > \|q-p\|_\infty$. According to Runge's Approximation Theorem (see, for example, [10, Theorem 13.7]), there is a polynomial g such that $|g-1| < \varepsilon/M$ on $r\bar{\mathbb{D}}$ and $|g| < \varepsilon/(2M)$ on C. Hence

$$|(g(z)-1)(q(z)-p(z))| < \varepsilon \quad (z \in r\bar{\mathbb{D}}), \quad |g(z)(q(z)-p(z))| < \frac{\varepsilon}{2} \quad (z \in C).$$

We fix $n_0 \in \mathbb{N}$ with

$$n_0 > \max\{\deg g(q-p), \deg p\}.$$

We set $z_k = z_{kn_0}$ and $w_k = g(z_k)(q(z_k)-p(z_k))$ $(k = 0, ..., n_0)$. Note that $|w_k| < \varepsilon/2$ $(k = 0, ..., n_0)$.

We now proceed according to an idea of Mortini [9]: According to [8, Theorem 1] there are finite Blaschke products $B_0, ..., B_{n_0}$ such that

$$B_k(z_k) = -1, \qquad B_k(z_j) = 1 \quad (j, k = 0, ..., n_0;\ j \neq k).$$

Now, let $m_0, ..., m_{n_0} \in \mathbb{N}$ and consider the function

$$h(z) = \sum_{j=0}^{n_0} w_j \frac{1-B_j(z)}{2} \left(\frac{1+\bar{z}_j z}{2}\right)^{m_j}.$$

We have, independent of the numbers m_j, that $h \in A(\mathbb{D})$, and $h(z_k) = w_k$ $(k = 0, ..., n_0)$. In particular $L_{K,n_0} h = g(q-p)$.

Now, let $\gamma > 0$ be such that the sets $C_k = \{z \in \mathbb{C} : |z - z_k| < \gamma\}$ $(k = 0, ..., n_0)$ are pairwise disjoint.

Since $|1 + \bar{z}_k z| < 2$ for $z \in \bar{\mathbb{D}} \setminus \{z_k\}$ we can choose $m_0, ..., m_{n_0}$ such that

$$\left|\frac{1+\bar{z}_k z}{2}\right|^{m_k} \leqslant \frac{1}{2^{k+1}} \qquad (z \in \bar{\mathbb{D}} \setminus C_k,\ k = 0, ..., n_0).$$

Then, for $k = 0, ..., n_0$ and $z \in C_k \cap \bar{\mathbb{D}}$

$$|h(z)| \leqslant |w_k| + \sum_{j=0, j \neq k}^{n_0} \frac{|w_j|}{2^{j+1}} \leqslant 2 \max\{|w_j| \colon j = 0, ..., n_0\},$$

and for $z \in \bar{\mathbb{D}} \setminus (\bigcup_{k=0}^{n_0} C_k)$

$$|h(z)| \leqslant \sum_{j=0}^{n_0} \frac{|w_j|}{2^{j+1}} \leqslant \max\{|w_j|: j=0, ..., n_0\}.$$

Together,

$$\|h\|_\infty \leqslant 2 \max\{|w_j|: j=0, ..., n_0\} < \varepsilon.$$

Set $f = p+h$. We have $\|f-p\|_\infty = \|h\|_\infty < \varepsilon$, and

$$L_{K, n_0} f - q = L_{K, n_0} p + L_{K, n_0} h - q = p + g(q-p) - q = (g-1)(q-p),$$

therefore $\|L_{K, n_0} f - q\|_r < \varepsilon$. ∎

Proof of Theorem 2. For $k \in \mathbb{N}$ consider

$$C_k = \{\exp(it): t \in [1/k, 2\pi - 1/k]\}.$$

Let K_k be the matrix in which the nodes in each row are equidistributed in C_k (with respect to the arc length). Then Theorem 1 applies to each K_k. Let U_k denote the corresponding set of functions $f \in A(\mathbb{D})$ for which $\{L_{K_k, n} f: n \in \mathbb{N}_0\}$ is dense in $H(\mathbb{D})$. Since each U_k is residual we can choose $f_0 \in \bigcap_{k \in \mathbb{N}} U_k$. Let $(r_k)_{k=1}^\infty$ be a sequence in $(0, 1)$ with limit 1, and let $(p_k)_{k=1}^\infty$ be a dense sequence of polynomials in $H(\mathbb{D})$. According to the definition of the sets U_k there is a strictly increasing sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}$ such that

$$\|L_{K_k, n_k} f_0 - p_k\|_{r_k} < \frac{1}{k} \qquad (k \in \mathbb{N}).$$

Set $n_0 = 0$. By selecting the rows with number n, $n_{k-1} < n \leqslant n_k$ of each matrix K_k and joining them, one obtains a matrix K satisfying (1). Moreover $f_0 \in U$: We have

$$L_{K, n_k} f_0 = L_{K_k, n_k} f_0 \qquad (k \in \mathbb{N}),$$

and therefore $\{L_{K, n} f_0: n \in \mathbb{N}_0\}$ is dense in $H(\mathbb{D})$. Finally, as we have mentioned in the introduction, $U \neq \varnothing$ implies that U is residual. ∎

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